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\*AN INFINITE CLASS OF PERIODIC SOLUTIONS OF

$$\ddot{x} + 2x^3 = p(t)$$

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1. Introduction. An important unsolved problem in the theory of non-linear oscillations is to establish the boundedness or unboundedness of the general solution of

$$\ddot{x} + g(x) = p(t),$$

(1.1) Performed in conjunction with RIAS, Baltimore, Md.

where dots denote differentiation with respect to  $t$ . When  $p(t)$  is periodic, we may seek periodic solutions. This search is interesting for its own sake, and of course leads us to special bounded solutions. In three previous papers (2,3,4) I have exhibited the equation

$$\ddot{x} + 2x^3 = e(t) \quad (1.2)$$

as tractable: on the assumption that  $e(t)$  is even and periodic, it was shown that the equation has an infinity of periodic solutions.

In this paper I show that

$$\ddot{x} + 2x^3 = p(t), \quad (1.3)$$

where  $p(t)$  is periodic, but is not necessarily even, also has an infinity of periodic solutions. The main result is:

THEOREM 2. Suppose that  $p(t)$  is continuously differentiable, that it has least period  $2\pi$  and that  $\int_0^{2\pi} p(t) dt = 0$ . Then for any positive integer  $m$ , (1.3) has an infinity of periodic solutions with least period  $2m\pi$ .

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Before we can attack this we need a criterion for a periodic solution of  $T$ , and this is to say we need a criterion for a topological transformation  $T$  associated with (1.3) to have a fixed point. (Cf. §2.) Our  $T$  has the property of being area-preserving, a property which is in many ways unwelcome since it precludes the use of the simpler methods appropriate to shrinking transformations. It does, however, allow of the following approach. Given a topological transformation  $U$  of the plane onto itself, we shall say a (Jordan) curve  $C$  is star-like for  $U$  if

- (i) the origin  $O$  does not lie on  $C$ ,
- (ii) any ray (half-line) through  $O$  meets  $C$  once at most, and
- (iii) any point  $P$  of  $C$  is transformed by  $U$  into a point on the same ray  $OP$ .

It is then evident from a figure (and we give the simple formal proof in §2) that we have

THEOREM 1. If  $U$  is an area-preserving transformation and  $C$  is a simple closed curve which is star-like for  $U$ , then there are at least 2 fixed points of  $U$  on  $C$ .

The greater part of this paper will be devoted to showing that, for any positive integer  $m$ , there are simple closed curves which are star-like for  $T^m$ . Once this is achieved Theorem 2 follows quickly.

The work gives a further application of the technique discussed in (4) for estimating certain partial derivatives. We shall assume the results (and the notation) in §§1 to 7 of (4); if the summary of (2) given in §2 of (4) is accepted, this paper is independent of (2). The work in (3) depends essentially on the evenness of  $c(t)$  and is irrelevant here.

2. Topological transformations of the plane. If  $x(t; a, b; 0)$  denotes the solution of (1.3) for which  $x(0) = a$ ,  $\dot{x}(0) = b$ , we define  $T(a, b)$  by

$$T(a, b) = (x(2\pi; a, b; 0), \dot{x}(2\pi; a, b; 0)).$$

We recall (see, for example Levinson (1)) that  $T$  is topological, that, since  $p(t)$  has period  $2\pi$ ,

$$T^m(a, b) = (x(2m\pi; a, b; 0), \dot{x}(2m\pi; a, b; 0)),$$

and that  $x(t; a, b; 0)$  has  $2m\pi$  as a period if and only if  $T^m(a, b) = (a, b)$ .

In this paper dashes will not be used as symbols of differentiation but only as labels, in particular as labels for transformed points and functions connected with them: for example, we shall usually write  $(a', b')$  for  $T^m(a, b)$ .

# Proof of Theorem 1. We note that a star-like simple closed curve must have the origin in its interior. Take polar coordinates with pole  $O$  and suppose  $C$  and  $UC$  have equations  $r = \gamma(\theta)$  and  $r = \gamma'(\theta)$ , each of  $\gamma(\theta)$  and  $\gamma'(\theta)$  being positive and continuous and having period  $2\pi$ . Then the difference of the areas inside  $C$  and  $UC$  is

$$0 = \frac{1}{2} \int_0^{2\pi} \{ \gamma^2(\theta) - \gamma'^2(\theta) \} d\theta.$$

Since the integrand is continuous and periodic there are at least two values of  $\theta$ , incongruent (mod  $2\pi$ ), for which  $\gamma(\theta) = \gamma'(\theta)$ .

LEMMA 1.  $T$  is area-preserving.

For any  $(a, b)$ , write  $u(t) = x(t; a, b; 0)$  and  $(a', b') = T(a, b)$ . Then the Wronskian,  $W(t)$  say, of those solutions  $y_1(t)$  and  $y_2(t)$  of

$$\ddot{z} + 6u^2(t)z = 0$$

for which  $z_1(0) = 1$ ,  $\dot{z}_1(0) = 0$  and  $z_2(0) = 0$ ,  $\dot{z}_2(0) = 1$ , is always 1, and hence  $\partial(a', b')/\partial(a, b) = W(2\pi) = 1$ .

3. The arrangement of the calculations. From here onwards we write  $(a', b') = T^m(a, b)$ ,  $m$  being taken as a constant. When we seek a curve which is star-like for  $T^m$ , we are seeking points whose coordinates satisfy

$$0.1 \quad S(a, b) \equiv ab' - a'b \equiv a \cdot b'(a, b) - a'(a, b) \cdot b = 1 \quad (3.1)$$

We shall determine arcs, parametrized either as  $(a(b), b)$  or  $(a, b(a))$ , composed of points whose coordinates satisfy (3.1). If we appeal to the standard implicit function theorem we must show either  $\partial S/\partial a$  or  $\partial S/\partial b$  different from 0, and in order to do this we estimate, in §5, the elements of the

Jacobian matrix  $\partial(a', b')/\partial(a, b)$ . Before this, in §4, we estimate  $\partial(a, b)/\partial(h_0, q_0)$ . After these preparations we can produce a star-like curve in §6 and prove Theorem 2.

The estimation of  $\partial(a, b)/\partial(h_0, q_0)$ , although of the type we met in (4), is sufficiently different to need to be given in some detail. Our other calculations are so light or so similar to those in (4) that we can abridge or suppress our discussion of them.

It will be clear that, if we chose, we could arrange to obtain more detailed information about periodic solutions of (1.3), especially if we varied the hypotheses on  $p(t)$  and considered relations such as  $m \leq Ak^2$  instead of taking  $m$  as a constant. However, the technique of estimation has been amply illustrated in (4); we therefore aim to show the application of Theorem 1 as simply as possible and do not strive after such detail.

4. The estimation of  $\partial(a, b)/\partial(h_0, \varphi_0)$ . Consider the solution  $x(t; a, b; 0)$  of (1.3) in the case when  $a \geq 0$ ,  $b \leq 0$ . Whether  $b^2 + a^*$  is large or not, this solution may also be specified by giving  $\varphi_0$ , the value of  $t$  at which the last maximum before  $t=0$  occurred ( $\varphi_0$  being 0 if  $b=0$ ), and  $h_0^*$ , the value of that maximum. It will be anticipated that if  $b^2 + a^*$  is large it is approximately equal to  $h_0^*$ ; in our error terms we shall most often use suitable powers of  $h_0$  alone, but it will sometimes be necessary to estimate in terms of products such as  $|b| h_0^{-3}$  so that we may later divide by  $b$  which could be small.

It will be useful to specialize some identities. (We give reference numbers from (4) in square brackets.) First, by putting  $t_0 = 0$  in [2.5] we find

$$h^*(t) = b^2 + a^* + 2 \int_0^t p(t') \dot{x}(t') dt',$$

and this gives, if we write  $a = h_0 \alpha$ ,

$$\begin{aligned} h_0^* &= h^*(\varphi_0) = b^2 + a^* + 2 \int_0^{\varphi_0} p(t) \dot{x}(t) dt \\ &= b^2 + a^* + 2 h_0 \int_{\alpha}^1 p\{t(h_0 \xi)\} d\xi. \end{aligned} \quad (4.1)$$

Rewriting this as

$$b^2 = h_0^* \left[ 1 - \alpha^* - 2 h_0^{-3} \int_{\alpha}^1 p\{t(h_0 \xi)\} d\xi \right]$$

and remembering that  $b \leq 0$ , we deduce that

$$-b = h_0^2 (1 - \alpha)^{\frac{1}{2}} \{f(\alpha; h_0, \varphi_0)\}^{\frac{1}{2}}, \quad (4.2)$$

where, as in [4.5],

$$f(\xi; h_0, \varphi_0) = 1 + \xi + \xi^2 + \xi^3 - 2 h_0^{-3} \cdot \frac{1}{1 - \xi} \int_{\xi}^1 p\{t(h_0 \xi')\} d\xi'.$$

Secondly, by writing  $\xi = \alpha$  in the expression [4.4] for  $\tau(h_0, \xi)$  we obtain

$$-h_0 \varphi_0 = \int_{\alpha}^1 (1-\xi)^{-\frac{1}{2}} \{f(\xi; h_0, \varphi_0)\}^{-\frac{1}{2}} d\xi. \quad (4.3)$$

Finally we observe that since  $\int_0^{2\pi} p(t) dt = 0$  there is no loss of generality in prescribing that  $p(0) = 0$ .

This simplifies some details; it also gives minor improvements in our estimations but these are inessential.

LEMMA 2. If  $b^2 + a^2$  is large, so is  $h_0$  and

$$b^2 + a^2 = h_0^4 + Q(1-\alpha) = h_0^4 + Q(1),$$

$$1-\alpha = Q(b^2 h_0^{-2}) = Q(|b| h_0^{-2}),$$

$$\sqrt{1-\alpha^2} = -b h_0^{-2} + Q(|b| h_0^{-6}). \quad (4.4)$$

LEMMA 3. For large  $h_0$ , in the case  $a \geq 0$ ,  $b \leq 0$ ,  $p(0) = 0$ ,

$$\frac{\partial(a, b)}{\partial(h_0, \varphi_0)} = \begin{pmatrix} (a - b\varphi_0)h_0^{-1} + Q(h_0^{-4}) & -b + Q(h_0^{-2}) \\ 2(b + a^3\varphi_0)h_0^{-1} + Q(h_0^{-3}) & 2a^3 + Q(h_0^{-2}) \end{pmatrix}.$$

By [4.8] we know that  $\partial \tau / \partial h_0 = Q(h_0^{-2})$  and

hence, by substitution in [4.7], that

$$\partial f / \partial h_0 = Q(h_0^{-5}),$$

where we have reduced our error term from  $Q(h_0^{-4})$  to

$Q(h_0^{-5})$  by using  $p(0) = 0$ . Differentiate (4.3) with respect to  $h_0$ ; we obtain, when we use (4.2),

$$-\varphi_0 = \frac{h_0^2}{b} \cdot \frac{\partial \alpha}{\partial h_0} + Q\{(1-\alpha)^{\frac{1}{2}} h_0^{-5}\}$$

and this gives

$$\partial a / \partial h_0 = (a - b\varphi_0)h_0^{-1} + Q(|b| h_0^{-6}).$$

We need this form with  $b$  as a factor in the error term for the immediately following work; a fortiori it gives the result enunciated.

If we now differentiate (4.1) with respect to  $h_0$  we obtain

$$2b \frac{\partial b}{\partial h_0} + 4a^3 \frac{\partial a}{\partial h_0} = 4h_0^3 - 2 \int_{\alpha}^1 p\{t(h_0, \xi)\} d\xi - 2h_0 \int_{\alpha}^1 \dot{p}\{t(h_0, \xi)\} \frac{\partial t}{\partial h_0} d\xi,$$

a further term on the right being suppressed since

$p\{t(h_0, \alpha)\} = p(0) = 0$ . Each of the terms involving an integral is  $Q\{(1-\alpha)h_0^{-1}\} = Q(|b|h_0^{-1})$ , and straightforward substitution gives

$$b \cdot \partial b / \partial h_0 = 2b(b + a^3 q_0) h_0^{-1} + Q(|b|h_0^{-1}).$$

If  $b \neq 0$ , our estimate of  $\partial b / \partial h_0$  follows; if  $b = 0$ , we see from the meaning of the symbols that  $\partial b / \partial h_0 = 0$  and  $b + a^3 q_0 = 0$ , and our estimate is again valid.

Very similarly, by differentiating (4.3) and (4.1) with respect to  $q_0$  we establish our estimates for  $\partial a / \partial q_0$  and  $\partial b / \partial q_0$ . We need to recall that by [4.9]

$$\partial t / \partial q_0 = 1 + Q(h_0^{-2})$$

and to observe that this gives  $\partial f / \partial h_0 = Q(h_0^{-1})$ .

5. The estimation of  $\partial(a', b') / \partial(a, b)$ . Suppose that  $(a, b)$  satisfies (3.1). We assume, as in §4, that  $a \geq 0$  and  $b \leq 0$  but no longer include this hypothesis in our enunciations; it implies, since  $(a', b')$  lies on the same ray through  $Q$  as  $(a, b)$ , that  $a' \geq 0$ ,  $b' \leq 0$ . If  $b^2 + a^4$  is suitably large we expect that the solution  $x(t; a, b; 0)$  will have only positive maxima and negative minima in  $0 \leq t \leq 2m\pi$ . ~~consequently~~  
We shall show it is consistent with

our previous notation to write  $q_0$  and  $q_{4n}$  for the values of  $t$  at which the maxima preceding (or at)  $t=0$  and  $t=2m\pi$  occur. This notation implies that the solution describes  $n$  cycles in  $q_0 \leq t \leq q_{4n}$ . Evidently

$$\frac{\underline{\partial}(a', b')}{\underline{\partial}(a, b)} = \frac{\underline{\partial}(a', b')}{\underline{\partial}(h_{4n}, q_{4n})} \cdot \frac{\underline{\partial}(h_{4n}, q_{4n})}{\underline{\partial}(h_0, q_0)} \cdot \frac{\underline{\partial}(h_0, q_0)}{\underline{\partial}(a, b)}.$$

Once we have verified that  $q_{4n}$  is defined and have shown that  $n$  is not too large compared with  $h_0$  we know how to estimate the matrices on the right-hand side.

We shall continue to assume  $p(0)=0$  but no longer include this hypothesis in our enunciations.

LEMMA 4. If  $(a, b)$  satisfies (3.1) and  $h_0$  is large,  
then  $q_{4n}$  is defined and there is a constant  $K$  such that  
 $n < Kh_0$ . Further,

$$a' = a \{1 + Q(h_0^{-2})\},$$

$$b' = b \{1 + Q(h_0^{-2})\}$$

and  $q_{4n} - 2m\pi = q_0 + Q(h_0^{-4}).$

We know that when  $\rho$  is given there is an  $R = R(\rho)$  with the property that if  $h_0 > R$  and  $2\nu \leq \rho h_0^2$  then  $q_{4\nu}$  is defined. For definiteness take  $\rho = 1$ . By an evident modification of [2.8],

$$q_{4\nu} - q_0 = 2\nu(\nu-1)h_0^{-1} + Q(\nu^2 h_0^{-5}),$$

the right-hand side of which exceeds  $2m\pi$  if  $h_0$  is large and  $\nu > \frac{1}{2} h_0^3$ . This is to say that, for large  $h_0$ , a  $q_{4\nu}$  is defined such that

$$q_{4\nu} > 2m\pi + q_0 > 2m\pi.$$

Hence  $q_{4n}$  is definable and we have at once  $2n < h_0^3$ .

From

$$2m\pi > \varphi_{n_n} - \varphi_n = 2\pi(n-1)h_0^{-1} + O(nh_0^{-2})$$

it follows that if we write  $K = 2m/\omega$  we have  $n < Kh_0$  for large  $h_0$ .

We recall that  $h_{n_n}^* = h_0^* + O(n) = h_0^* + O(h_0)$ .  
Write  $a'^2 = (1+\eta)a^2$  and  $b'^2 = (1+\eta)b^2$ ; then  

$$b'^2 + a'^2 = h_{n_n}^* + O(h_{n_n})$$

$$= h_0^* + O(h_0) = b^2 + a^2 + O(h_0),$$
 that is,  $\eta(b^2 + a^2) + \eta(1+\eta)a^2 = O(h_0)$ .

Whether  $\eta \geq 0$  or  $-1 < \eta < 0$ , the terms on the left-hand side have the same sign and we see that  $\eta = O(h_0^{-3})$ , which gives the first and second estimates.

By use of (4.4) we see that

$$\begin{aligned} \sqrt{(1-\alpha'^2)} &= -b'h_{n_n}^{-2} + O(|b'|h_{n_n}^{-6}) \\ &= -bh_0^{-2} + O(h_0^{-3}) = \sqrt{(1-\alpha^2)} + O(h_0^{-3}). \end{aligned}$$

It is clear that (4.3) gives

$$\varphi_0 = -h_0^{-1} \int_{\alpha}^1 (1-\xi^2)^{-\frac{1}{2}} d\xi + O(h_0^{-5}),$$

and that  $\varphi_{n_n} - 2m\pi$  may be written for  $\varphi_0$ ,  $h_{n_n}$  for  $h_0$  and  $\alpha'$  for  $\alpha$  to give

$$\begin{aligned} \varphi_{n_n} - 2m\pi &= -h_{n_n}^{-1} \int_{\alpha}^1 (1-\xi^2)^{-\frac{1}{2}} d\xi - h_{n_n}^{-1} \int_{\alpha'}^{\alpha} (1-\xi^2)^{-\frac{1}{2}} d\xi + O(h_{n_n}^{-5}) \\ &= \varphi_0 + O(h_0^{-5}) + O\{h_0^{-1} |\sqrt{(1-\alpha^2)} - \sqrt{(1-\alpha'^2)}|\} \\ &= \varphi_0 + O(h_0^{-4}). \end{aligned}$$

LEMMA 5. Under the hypotheses of Lemma 4

$$\frac{\partial(\alpha', b')}{\partial(h_{n_n}, \varphi_{n_n})} = \begin{pmatrix} (a-b\varphi_0)h_0^{-1} + O(h_0^{-3}) & -b + O(h_0^{-3}) \\ 2(b+a^2\varphi_0)h_0^{-1} + O(h_0^{-2}) & 2a^2 + O(1) \end{pmatrix}.$$

LEMMA 6. If  $n < Kh_0$  then, for large  $h_0$ ,

$$\frac{\partial(h_{4n}, q_{4n})}{\partial(h_0, q_0)} = \begin{pmatrix} 1 + o(h_0^{-2}) & o(h_0^{-1}) \\ -2\pi n h_0^{-2} + o(h_0^{-1}) & 1 + o(h_0^{-1}) \end{pmatrix}.$$

This is obtained at once by the methods of (4).

LEMMA 7. For large  $h_0$ ,

$$\frac{\partial(h_0, q_0)}{\partial(a, b)} = \frac{1}{2} h_0^{-3} \begin{pmatrix} 2a^2 + Q(h_0^{-1}) & b + Q(h_0^{-2}) \\ -2(b + a^2 q_0) h_0^{-1} + Q(h_0^{-3}) & (a - b q_0) h_0^{-1} + Q(h_0^{-2}) \end{pmatrix}.$$

LEMMA 8. Under the hypotheses of Lemma 4,

$$\frac{\partial(a', b')}{\partial(a, b)} = \begin{pmatrix} 1 + 2\pi n a^2 b h_0^{-5} + o(h_0^{-4}) & \pi n b^2 h_0^{-5} + o(h_0^{-4}) \\ -4\pi n a^4 h_0^{-5} + o(1) & 1 - 2\pi n a^2 b h_0^{-5} + o(h_0^{-4}) \end{pmatrix},$$

$$\partial S / \partial a = -2\pi n a^3 h_0^{-5} (2a^2 + b^2) + o(h_0),$$

$$\partial S / \partial b = -\pi n b h_0^{-5} (2a^2 + b^2) + o(1).$$

6. A curve star-like for  $T^m$ . From (4) we know that when  $m$  and  $n$  are suitably restricted there is a unique root of

$$\varphi_{2n}(h_0, 0) = m\pi, \quad (6.1)$$

in particular, when  $m$  is fixed there is a unique root for each large  $n$ . It will be clear that by trivial modifications of the argument we can show that there is a unique root of

$$\varphi_{4n}(h_0, 0) = 2m\pi. \quad (6.2)$$

When we consider an even  $e(t)$  these two unique roots will coincide. For a general  $p(t)$  the equations (6.1) and (6.2) have different roots, and it is (6.2) which is relevant; we shall write  $h_0^*$  for the root of (6.2).

Write  $P_0$  for the point  $(h_0^*, 0)$  of the  $(a, b)$ -plane. Then (6.2) implies that all the stationary points of  $x(t; h_0^*, 0; 0)$ , or  $x(t; P_0; 0)$  say, are positive maxima or negative minima and that there are  $n$  of each in  $0 \leq t < 2m\pi$ . Further,  $P_0$  is the only point on the positive  $a$ -axis with these properties.

Similarly, if we specify a solution of (1.3) as  $x(t; 0, -h_1^*; q_1)$  we can consider the function  $\varphi_{n+1}(h_1, q_1)$  and, again by trivial modifications of our previous arguments, can show that

$$\varphi_{n+1}(h_1, 0) = 2m\pi$$

has a unique root, say  $h_1^*$ . Write  $P_1$  for the point  $(0, -h_1^*)$ . Then  $P_1$  and  $T^m P_1$  both lie on the negative half of the  $b$ -axis and  $P_1$  is uniquely characterized by the properties of  $x(t; P_1; 0)$ . In exactly the same way we find points  $(-h_2^*, 0)$ ,  $(0, h_2^*)$ , say  $P_2$  and  $P_3$ , whose transforms by  $T^m$  lie on the same half-axis as the initial points. Finally we observe that if  $h_n^*$  is the root of

$$\varphi_{n+n}(h_n, 0) = 2m\pi$$

the point  $(h_n^*, 0)$ ,  $P_n$  say, has the properties of  $P_0$  and we obtain  $P_n = P_0$ .

These remarks can be brought into a form suitable for application if, as before, we write  $n = m\kappa + s$  and think of  $\kappa$  as a large integer. We have

LEMMA 9. Suppose  $m$  is fixed and that  $s$  is chosen: if  $m = 1$ ,  $s$  is to be 0; if  $m > 1$ ,  $s$  is to be prime to  $m$  and  $0 < s < m$ . Suppose that the half-axes are numbered ( $i = 0, 1, 2, 3$ ) clockwise from the positive  $a$ -axis. Then for each large integer  $\kappa$  there is a point

$P_i = P_i(k)$  on the  $i$ -th half-axis such that

- (i)  $T^m P_i$  and  $P_i$  lie on the same half-axis,
- (ii) all the stationary points of  $x(t; P_i; 0)$  are positive maxima or negative minima, and there are  $m\kappa + s$  of each in  $0 \leq t < 2m\pi$ , and
- (iii)  $P_i$  are uniquely determined by (i) and (ii).

Further,

$$h_i^* = (k + s/m)\omega + Q(k^{-2}).$$

Proof of Theorem 2. For large  $k$ ,  $\partial S / \partial a \neq 0$  at  $P_i$  and so for a range of  $b \leq 0$  we have an arc parametrized  $(a(b), b)$  whose points satisfy (3.1). At least on a sub-arc  $b^2 + a^*$  is large, with  $a^* \geq b^2$ , and the hypotheses of Lemma 4 (and hence of Lemma 8) are fulfilled. The integer  $n$  whose existence is guaranteed by Lemma 4 will change continuously and hence have the constant value  $m\kappa + s$ . When we apply Lemma 8 we see that on the sub-arc  $\partial S / \partial a \neq 0$  and

$$\frac{da}{db} = - \frac{\partial S / \partial b}{\partial S / \partial a} = - \frac{b + \varepsilon(1)}{2a^2 + \varepsilon \{h_i(a, b)\}},$$

which gives  $d(b^2 + a^*) / db = \varepsilon(1)$ . It follows that, for large  $k$ , if the sub-arc does not already reach the curve  $a^* = b^2$  it can be uniquely continued, with the parametrization  $(a(b), b)$  until it does, and that  $b^2 + a^*$  remains large. Beyond this we can evidently continue with the parametrization  $(a, b(a))$  until we arrive at a point on the negative  $b$ -axis with the characteristic properties of  $P_i$ , that is at  $P_i$ .

If a ray through  $Q$  met this arc in two points we could, by continuously turning the ray, find a ray which touched the arc, contrary to the uniqueness of  $a(b)$  and  $b(a)$ . To see that the arc is star-like for  $T^m$  we need now only notice that

$\underline{Q}$  does not lie on the arc since  $b^2 + a^2$  remains large.

Similarly  $P_1$  can be joined to  $P_2$ ,  $P_2$  to  $P_3$  and  $P_3$  to  $P_4 = P_0$ . Write  $C = C(k)$  for the curve formed by these arcs, then  $C$  and  $T^m$  satisfy the hypotheses of Theorem 1. Hence we have, for each large  $k$ , at least two points fixed under  $T^m$ , say  $Q_1(k)$  and  $Q_2(k)$ , and this means that  $x(t; Q_1(k); 0)$  and  $x(t; Q_2(k); 0)$  have  $2m\pi$  as a period. Finally this must be the least period since each of these solutions has  $mk+s$  maxima in  $0 \leq t < 2m\pi$ .

COROLLARY. For any  $m$  and every sufficiently large  $k$ , there are at least two solutions of (1.3) with least period  $2m\pi$ , describing  $mk+s$  cycles in this period and with  $b^2 + a^2 = (k+s/m)^n \omega^2 + \underline{Q}(k)$ .

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